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Wilson loops in the Higgs phase of large-N field theories on the conifold

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Abstract: We study the quark-antiquark interaction in the large-N limit of the superconformal field theory on D3-branes at a Calabi-Yau conical singularity. We compute the Wilson loop in the $AdS_5 \times T^{11}$ supergravity background for the $\mathrm{SU}(2N) \times \mathrm{SU}(2N)$ theory. We also calculate the Wilson loop for the Higgs phase where the gauge group is broken to $\mathrm{SU}(N) \times \mathrm{SU}(N) \times \mathrm{SU}_D(N)$. This corresponds to a two center configuration with some of the branes at the singularity and the rest of them at a smooth point. The calculation exhibits the expected Coulomb dependence for the interaction. The angular distribution of the BPS states is different than the one for a spherical horizon.

Keywords: AdS-CFT Correspondance, D-branes.

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Contents

1. Introduction

The long suspected correspondence [1, 2] between string theory and the limit in which the number of colors of a gauge field theory is taken to infinity was first explicitly constructed by Maldacena [3], between a four dimensional $N = 4$ supersymmetric gauge theory, and type-IIB string theory compactified on $AdS_5 \times S^5$. According to the more precise statements of the conjecture elaborated in [4] and [5], correlation functions of gauge theory operators can be explicitly calculated from type-IIB string theory on $AdS_5 \times S^5$. From the arguments in [5], type-IIB string theory on $AdS_5 \times X_5$, with X_5 a five dimensional Einstein manifold bearing five-form flux, is supposed to be dual to a four dimensional conformal field theory. Generalizations of the original idea have been constructed considering the sphere of the maximally supersymmetric case, S^5, divided by the action of some finite group [6]–[9]. The field theory thus obtained corresponds to the infrared limit of the worldvolume theory on D3-branes at an orbifold singularity (the case of 7-branes at an orientifold singularity has also been considered in [10]–[13]). Another proposal of duality arises from the study of D3-branes at Calabi-Yau singularities [14]–[20]. The connection between compactification on Einstein manifolds and the metric of D3-branes placed at a Calabi-Yau singularity pointed out in [14], was enlightened by Klebanov and Witten, who constructed the field theory side for a smooth Einstein manifold with local geometry different from the sphere, $X_5 = T^{11}$ [15]; their proposal is that the type-IIB string theory compactified on $AdS_5 \times T^{11}$ should be dual to a certain large-N superconformal field theory with $N = 1$ supersymmetry in four dimensions, which turns out to be a non trivial infrared fixed point, with gauge group $\mathrm{SU}(N) \times \mathrm{SU}(N)$. The field theory is constructed locating a collection of N coincident D3-branes at a conical singularity of a non compact Calabi-Yau manifold (see also [19] for an exhaustive study of non-spherical horizons, and the corresponding field theories). In a subsequent paper [20], these authors also considered the posibility that the branes are extracted from the conifold point. Assuming that the branes are all at the same point, the low energy effective field theory will be the $N = 4$ supersymmetric $\mathrm{SU}(N)$ gauge theory, which is again an infrared fixed point.

In this paper, we will consider an intermediate situation in between these two theories, in order to clarify the symmetry breaking, corresponding to a two center configuration, where some of the branes are still located at the conifold point, while the rest have been moved to some smooth point. We will study some details of this configuration describing the Higgs phase of the field theory of D3-branes at the conifold. Applying the construction for the Wilson loops introduced by Maldacena in [21] to an $AdS_5 \times T^{11}$ background, we will explicitly calculate the energy for a quark-antiquark pair, separated by a distance L. As we will see, the form of the calculation we present also reduces naturally to the possibility that all branes are located at the conifold point, originally considered in [15], where we can also reproduce the expected conformal behaviour for the energy of the quark-antiquark pair. The expression for the energy, obtained minimizing the area of a string worldsheet in the $AdS_5 \times T^{11}$ background, allows also the possibility to understand configurations of vanishing energy.

In section 2, after a small review on the conifold and its properties, we will, following [15], study the $\mathrm{SU}(N) \times \mathrm{SU}(N)$ gauge theory broken to the Higgs phase $\mathrm{SU}(N) \times \mathrm{SU}(N) \times \mathrm{SU}_D(N)$. The Higgs phase is explored in section 3 through the calculation of the Wilson loop in $AdS_5 \times T^{11}$, for a configuration of N branes at the conifold point, and N branes located at a smooth point. The general solution we construct allows the possibility to consider the unbroken phase, with all branes located at the conifold. In section 4 we present some concluding remarks and possible implications of the present work, as well as some immediate steps further.

2. Large-N field theories and conifolds

In this section we will review some basic properties of a six dimensional Calabi-Yau manifold Y_6 with a conical singularity and study the Higgs branch for a particular configuration of D3-branes on $M_4 \times Y_6$. According to [15] and [20] keeping some of the branes at the conifold point implies $N = 1$ supersymmetry. The configuration we will consider breaks the gauge group to $\mathrm{SU}(N) \times \mathrm{SU}(N) \times \mathrm{SU}_D(N)$ and has $N = 1$ supersymmetry.

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2.1 Conifold spaces

Let Y_6 be a six dimensional manifold smooth apart from an isolated conical singularity, and of vanishing first Chern class. Y_6 can be described by a quadric in $\mathbb{C}^4 \sim \mathbb{R}^8$,

$$\mathcal{C} \equiv \sum_{A=1}^{4} (w^A)^2 = 0 \,. \tag{2.1}$$

The singularity is located at $w^A = 0$ for which $\mathcal{C} = 0$ and $d\mathcal{C} = 0$. Let $\mathcal{B}$ be the base of the cone. $\mathcal{B}$ is given by the intersection of (2.1) and a sphere of radius ρ in $\mathbb{C}^4$,

$$\sum_{A=1}^{4} |w^A|^2 = \rho^2 \,. \tag{2.2}$$

In order to examine the topology of $\mathcal{B}$ we rewrite equations (2.1) and (2.2) in terms of the real and imaginary parts of w^A, $w^A = x^A + iy^A$, with $x^A, y^A \in \mathbb{R}$,

$$\begin{aligned} x \cdot x &= \frac{1}{2}\rho^2 \,, \\ y \cdot y &= \frac{1}{2}\rho^2 \,, \\ x \cdot y &= 0 \,. \end{aligned} \tag{2.3}$$

The first equation describes an S^3. The other two define a trivial S^2 fiber over S^3. Thus, the base $\mathcal{B}$ is topologically $S^2 \times S^3$.

We want Y_6 to have vanishing first Chern class; however, the metric on a cone is Ricci flat if and only if the metric on its base is Einstein; thus, $\mathcal{B}$ must be a 5 dimensional Einstein space, $\mathcal{R}_{\mu\nu} = 4g_{\mu\nu}$, with the topology of $S^2 \times S^3$.

Let us now consider metrics on manifolds $T^{p,q}$[1] which are fiber bundles over $S^2 \times S^2$, with (p, q) U(1) fiber, whose metric is [22]

$$\begin{aligned} d\Sigma^2_{pq} = {} & \lambda^2(d\psi + p\cos\theta_1 d\phi_1 + q\cos\theta_2 d\phi_2)^2 + \Lambda_1^{-1}(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) + \\ & + \Lambda_2^{-1}(d\theta_2^2 + \sin^2\theta^2 d\phi_2^2), \end{aligned} \tag{2.4}$$

where $0 \le \theta_i < \pi$ and $0 \le \phi_i < 2\pi$ are spherical coordinates and $0 \le \psi < 4\pi$ is the coordinate on the U(1) fiber. For some choices of $\Lambda_1, \Lambda_2, \lambda$ these metrics are Einstein. For two choices of (p, q), namely $(1, 1)$ and $(1, 0)$, the fiber bundles are $S^2 \times S^3$. It was shown in [22] that only $T^{1,1}$ is compatible with a Kähler structure on the cone. In other words, these two Einstein metrics on the base give rise to two Ricci flat metrics on the cone, but only taking $T^{1,1}$ as the base we get a cone which is a limit of a Calabi-Yau metric. In this case Y_6 will admit Killing spinors and thus, preserve supersymmetry. More precisely, Y_6 is a manifold of SU(3) holonomy, therefore it

[1] The notation differs slightly from that in [22], where these spaces are denoted by $N_{p,q}$.

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preserves a quarter of the supersymmetries of the original theory. This implies that the four dimensional theory will have 8 supercharges and the theory will be $N = 1$ supersymmetric.

Redefining the radial coordinate as $r \equiv \sqrt{3/2}\,\rho^{2/3}$ we can write the Y_6 metric in the conical form,

$$ds_6^2 = dr^2 + r^2 d\Sigma_{11}^2\,, \tag{2.5}$$

where the metric of the base is

$$d\Sigma_{11}^2 = \frac{1}{9}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2 + \frac{1}{6}(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) + \frac{1}{6}(d\theta_2^2 + \sin^2\theta^2 d\phi_2^2)\,. \tag{2.6}$$

The Kähler potential, K, for the conifold metric (2.5) should be invariant under the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ that acts on the base. In [22] the authors found $K = (\sum_{A=1}^{4}|w^A|^2)^{2/3}$.

For T^{11} the U(1) fiber is symmetrically embedded in the two SU(2)'s. The U(1) generator is the sum $\frac{1}{2}\sigma_3 + \frac{1}{2}\tau_3$ of the SU(2)'s generators and T^{11} is the coset space

$$T^{11} = \frac{\mathrm{SU}(2)\times\mathrm{SU}(2)}{\mathrm{U}(1)} = \frac{S^3\times S^3}{\mathrm{U}(1)}\,. \tag{2.7}$$

This space can also be thought of as a smooth deformation of a blown-up $S^5/\mathbb{Z}_2$ orbifold. The isometry group of the orbifold is $\mathrm{SO}(4)\times\mathrm{SO}(2)$, which is the same as for T^{11}, and an appropiate blow up of the orbifold singularities leads to a manifold with the same topology as T^{11}.

2.2 The field theory on D3-branes at the conifold

In [15] the authors studied the theory obtained when N D3-branes are placed on a conifold singularity. They considered a decomposition of ten dimensional space given by $M_4\times Y_6$, where M_4 is four dimensional Minkowski space, and Y_6 is a Calabi-Yau manifold with a conical singularity, of the form considered in section 2.1. Let the singularity be at a point P on the conifold. If we place N parallel D3-branes on $M_4\times P$, the resulting ten dimensional metric will be

$$ds^2 = H(r)^{-1/2}[-dt^2 + dx^2] + H(r)^{1/2}[dr^2 + r^2 g_{ij}dx^i dx^j]\,. \tag{2.8}$$

where the harmonic function is $H(r) = 1 + L^4/r^4$, with $L^4 = 4\pi g_s N(\alpha')^2$.

In the near horizon limit this metric becomes $AdS_5\times T^{11}$, where T^{11} has been defined in (2.7), with the metric (2.6). The infrared limit of the gauge theory on the D3-branes is $N = 1$ supersymmetric with gauge group $\mathrm{SU}(N)\times\mathrm{SU}(N)$ and chiral superfields A_i, B_j $i,j = 1,2$. A_i transforms as $(\mathbf{N},\bar{\mathbf{N}})$ and B_i as $(\bar{\mathbf{N}},\mathbf{N})$ under $\mathrm{SU}(N)\times\mathrm{SU}(N)$. The theory has a non-renormalizable superpotential,

$$W = \frac{\lambda}{2}\epsilon^{ij}\epsilon^{kl}A_iB_kA_jB_l\,. \tag{2.9}$$

The moduli space of vacua of the theory is the conifold $\mathcal{C}$ defined in (2.1). The eigenvalues of A_i, B_j are related to the the positions of the branes.

The Higgs phase. Symmetry breaking arises when the branes are moved away from the conifold singularity, away from each other or when the singularity is resolved or deformed [15]. In the present paper we are interested in symmetry breaking obtained by moving the branes away from the conifold singularity. The case when all the branes are moved to a smooth point has been thoroughly studied. If the gauge symmetry is broken to the diagonal group [20], $\mathrm{SU}(N) \times \mathrm{SU}(N) \to \mathrm{SU}_D(N)$, the fields will transform in the adjoint representation of the group. One of them can be Higgsed away, leaving three adjoints with the superpotential

$$W = \mathrm{Tr}(\Phi_1[\Phi_2, \Phi_3])\,. \tag{2.10}$$

This theory flows in the infrared to $N = 4$ supersymmetric Yang Mills, as expected since now the D3-branes are located on a smooth point of the manifold.

But we want to investigate a different pattern of symmetry breaking. We start with $2N$ branes at the conifold singularity, i.e., with a $\mathrm{SU}(2N) \times \mathrm{SU}(2N)$ theory with two copies of matter in the $(\mathbf{2N}, \overline{\mathbf{2N}})$ and $(\overline{\mathbf{2N}}, \mathbf{2N})$ and a superpotential as described in equation (2.9). We will consider the theory obtained when we move half of the branes, N from each group, to a smooth point $\vec{r} = \vec{r}_0$ away from the singularity and leave the other half at the singularity $r = 0$. The gauge group is then broken to $\mathrm{SU}(N) \times \mathrm{SU}(N) \times \mathrm{SU}_D(N)$. The states charged under the $\mathrm{SU}_D(N)$ and either of the $\mathrm{SU}(N)$'s correspond to strings stretching from the branes in the singularity to the branes at $r = r_0$, away from the conifold point. These are the W bosons responsible for the symmetry breaking. The fields left will transform as $(\mathbf{N}, \bar{\mathbf{N}}, \mathbf{1})$, $(\bar{\mathbf{N}}, \mathbf{N}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{1}, \mathbf{N^2} - \mathbf{1})$ under the Higgsed gauge group. Note that this pattern of symmetry breaking, as opposed to the one considered in [20], does not alter the amount of supersymmetry since the harmonic function is still singular at the conifold point $r = 0$.

The process of moving branes out of the conifold could be done in steps, i.e., successively moving groups of M_i branes. In this case the gauge group at the smooth point will not be $\mathrm{SU}_D(2N)$ but a collection of independant $\mathrm{SU}_D(M_i)$'s. This can be understood as a consequence of the different pattern of symmetry breaking, since the intermediate steps imply the Higgsing of more states.

3. Wilson loops in non spherical horizons

The conjectured equivalence proposed by Maldacena between the large-N limit of four dimensional Yang-Mills with $N = 4$ supersymmetry, and ten dimensional supergravity, allows the possibility to predict a number of interesting phenomena of field theories in the large-N limit. In [23] and [21], a method to calculate the expectation value of the Wilson loop operator in the large-N limit of field theories was proposed (see [24] for a recent review, and a collection of references on the subsequent cases studied in the literature). In this work, we will apply the technique introduced in [21] to calculate the area of a fundamental string worldsheet in the $AdS_5 \times T^{11}$ background, and understand the energy, of a quark-antiquark pair, in the Higgs phase of the theory of branes at the conifold described in the previous section. The calculation we present also reproduces naturally the unbroken Higgs phase, where all branes are located at the conifold singularity.

3.1 Multicenter configurations

The original configuration first studied by Maldacena in [3] was a collection of N parallel D-3branes, located at some point in their transverse space. The possibility to consider separated groups of D3-branes, that for large N would also reproduce some anti-de Sitter geometry, was also first mentioned in [3]. Multicenter D3-brane solutions were originally considered in [25], and studied in the background of a Calabi-Yau threefold with a conical singularity in [20]. The multicenter configuration corresponds to the harmonic function

$$H(\vec{r}) = 1 + \sum_a \frac{4\pi g_s N_a \alpha'^2}{|\vec{r} - \vec{\hat{r}}_a|^4}\,, \tag{3.1}$$

where $\vec{r}$ denotes the coordinates transverse to the worldvolume of the D3-branes, $\vec{r} = \{y^1, \ldots, y^6\}$, on the Calabi-Yau manifold, and $\vec{\hat{r}}_a$ are the positions of the D3-branes. We will measure distances, with respect to the conifold point, in terms of $r = \sqrt{y^i y^i}$. As in [3], the limit where $\alpha' \to 0$, keeping r/α' finite and all N_a large, will correspond to the Higgs phase of an SU(N) gauge theory, with $N = \sum_a N_a$, where the gauge group has been broken to $\prod_a \mathrm{SU}(N_a)$. However, at large values of r compared to the distances between the groups of D3-branes, which represent the Higgs vacuum expectation values, the multicenter configuration reproduces the anti-de Sitter solution corresponding to SU(N). As long as we keep some branes at the conifold point, as in section 2, the theory will be $N = 1$ supersymmetric.

Following the construction in section 2, we will study the Wilson loop for a Higgs phase of the $N = 1$ supersymmetric field theory of D3-branes at the conifold. The multicenter configuration will correspond to a set of N D3-branes located at some smooth point $\vec{r} = \vec{\hat{r}}_0$, and a collection of N of them at the conifold point, so that the final gauge group is $\mathrm{SU}(N) \times \mathrm{SU}(N) \times \mathrm{SU}_D(N)$, as in section 2. In the near horizon limit ($\alpha' \to 0$ but r/α' finite), the harmonic function becomes

$$H(\vec{y}) = \frac{4\pi g_s N \alpha'^2}{r^4} + \frac{4\pi g_s N \alpha'^2}{|\vec{r} - \vec{\hat{r}}_0|^4}\,. \tag{3.2}$$

For the harmonic function (3.2), there is a clear limit, corresponding to $r \gg \hat{r}_0$, where the expression for H simplifies to that of a single collection of $2N$ D3-branes located at the conifold point, $H = 2 \cdot 4\pi g_s N \alpha'^2 / r^4$. We will also address this limit for our calculation at the end of this section. From a field theory point of view, it implies that the mass scale of the symmetry breaking of $\mathrm{SU}(2N) \times \mathrm{SU}(2N)$ to $\mathrm{SU}(N) \times \mathrm{SU}(N) \times \mathrm{SU}_D(N)$, proportional to the distance in between the two sets of branes, r_0, is much smaller than the energy scale of the quark-antiquark potential, so that the interaction will correspond to the unbroken $\mathrm{SU}(N) \times \mathrm{SU}(N)$ phase.

We are interested in the harmonic function (3.2) describing the configuration with two centers. However, from a calculational point of view, this function is extremely involved. In fact, a similar calculation for a Higgs phase of branes at two different positions, but for the $AdS_5 \times S^5$ background, was considered in [26]. The authors of [26] used a symmetric configuration in order to simplify the form of the harmonic function. Our work requires a similar choice of symmetric configuration.[2] The main difficulty arises due to the fact that H contains two terms for a two center configuration, and that it is H^{-1} that enters the Nambu-Goto action in the calculation of the Wilson loop. Hence, the most convenient possibility is to restrict the calculation to the region where

$$|\vec{r}| = |\vec{r} - \vec{r}_0| \,. \tag{3.3}$$

This constraint corresponds to a symmetric configuration where the minimum of the Wilson loop is located at the vertex of the distinct angle of an isosceles triangle. The two sets of branes (those located at the conifold, and those at the smooth point) are then at the two other vertices, so that the two equal sides of the triangle will correspond to the distance to the loop.

As we will see next, the Wilson loop can be calculated exactly for this symmetric configuration.

3.2 Geodesics of the symmetric configuration

In order to calculate Wilson loops according to the prescription in [21], the Nambu-Goto action of a fundamental string should be minimized in a relevant supergravity background. We should then consider the action for the string worldsheet,

$$S = \frac{1}{2\pi\alpha'} \int d\sigma d\tau \sqrt{\det G_{MN} \partial_\alpha X^M \partial_\beta X^N} \,, \tag{3.4}$$

in the $AdS_5 \times T^{11}$ background, where the G_{MN} metric is

$$ds_{10}^2 = H^{-1/2}(r)(-dt^2 + dx_i dx^i) + H^{1/2}(r)(dr^2 + r^2 d\Sigma_{11}^2) \,. \tag{3.5}$$

Defining $U = r/\alpha'$, this metric becomes

$$ds_{10}^2 = \alpha' \left[H^{-1/2}(U)(-dt^2 + dx_i dx^i) + H^{1/2}(U)(dU^2 + U^2 d\Sigma_{11}^2) \right] \,, \tag{3.6}$$

[2]Another possibility to describe the harmonic function of a multicenter configuration in an $AdS_5 \times T^{11}$ background is to use an expansion for H in terms of spherical harmonics, as in [20]. However, again from a calculational point of view, the harmonic function leads to extremely complicated expansions for the solutions, which can only be calculated as a power series.

so that now the harmonic function describing the multicentered configuration is

$$H(U) = \frac{R^4}{U^4} + \frac{R^4}{|\vec{U} - \vec{r}_0|^4}\,, \tag{3.7}$$

where we have introduced $R^4 \equiv 4\pi g_s N$, and the rescaling $\vec{r}_0 \equiv \vec{\hat{r}}_0/\alpha'$ has also been employed, so that $\vec{r}_0$ is kept constant.

Now, following [21], we can introduce a relative angle between the quarks, by giving expectation values $\vec{\Phi}_1$ and $\vec{\Phi}_2$ to two U(1) factors in the global gauge group. If the angles are defined as $\vec{\theta}_i = \vec{\Phi}_i/|\vec{\Phi}|$, the string worldsheet will be located, at the end of the Wilson loop corresponding to the position of the massive quark, at $U = \infty$ and a point $\vec{\theta}_1$ on the T^{11} space, and at $U = \infty$ but a point $\vec{\theta}_2$ on T^{11} for the other end of the loop, corresponding to the position of the other massive quark. However, as seen in section 2.1, the topology of T^{11} is quite different from that of S^5. The SO(6) isometry group of S^5 was responsible that the string joining $\vec{\theta}_1$ and $\vec{\theta}_2$ in [21], would simply lie on a great circle of the sphere. The SO(4) $\times$ SO(2) isometry group of T^{11} also allows this possibility: the Wilson loop can be taken to extend along a great circle in S^2, or in S^3; this simply corresponds to setting all angles, but one, to constants, in the metric describing T^{11}. The loop constructed this way clearly reproduces that studied in [21] for strings located at different points in S^5, up to a numerical coefficient related to the constraints defining the T^{11} metric. However, SO(4) $\times$ SO(2) does not exclude other possibilities. The Wilson loop can be taken to lie on a path parametrized by two, or even three, different coordinates on T^{11}, and which cannot be described in terms of a single angle through a rotation. For definiteness, we will choose the path joining $\vec{\theta}_1$ and $\vec{\theta}_2$ to be parametrized by ψ and ϕ_1 (the two other cases with two angles, (ψ, ϕ_2) and (ϕ_1, ϕ_2), or the slightly more involved of a trajectory requiring the set (ψ, ϕ_1, ϕ_2), can be similarly treated). Then, setting ϕ_2 = constant, and choosing the convenient values $\theta_1 = \theta_2 = 0$, the action for a static configuration becomes

$$S = \frac{T}{2\pi}\int dx \sqrt{H^{-1}(U) + (\partial_x U)^2 + \frac{U^2}{9}(\partial_x\psi)^2 + \frac{U^2}{9}(\partial_x\psi)(\partial_x\phi_1) + \frac{U^2}{6}(\partial_x\phi_1)^2}\,, \tag{3.8}$$

where we have identified the space worldsheet coordinate σ with one of the spatial coordinates on the worldvolume of the D3-branes.

The Euler-Lagrange equations for this action are, in general, quite hard to solve for the two center configuration we are interested in, unless a symmetric constraint as the one described in (3.1) is chosen. We will work in such a configuration, so that our solutions will represent the geodesics satisfying condition (3.3).

As the action does not depend on x explicitly, the associated energy

$$\frac{U^4/2}{\sqrt{H^{-1}(U) + (\partial_x U)^2 + U^2/9(\partial_x\psi)^2 + U^2/9(\partial_x\psi)(\partial_x\phi_1) + U^2/6(\partial_x\phi_1)^2}} \tag{3.9}$$

will be a conserved quantity. Besides, the action is also cyclic in ψ and ϕ_1, so that the associated angular momenta, $\partial L/\partial(\partial_x\psi)$ and $\partial L/\partial(\partial_x\phi_1)$ (where the lagrangian L is the square root in (3.8)), will also be constants. The value of these constants can be fixed in terms of the minimum of the Wilson loop. If U_0 is the distance along the slice transversal to the direction defined by the position of the conifold point, and the N D3-branes at $\vec{r}_0$, outside the singularity,[3] then, in the symmetric configuration of section 3.1, the minimum will correspond to a radial coordinate $U_{\rm min} = \sqrt{U_0^2 + (r_0/2)^2}$, measured with respect to any of the two sets of D3-branes. Then, solving for $\partial_x\psi|_{U_{\rm min}}$ and $\partial_x\phi_1|_{U_{\rm min}}$, the constants can be fixed to

$$\begin{aligned}\frac{\partial L}{\partial(\partial_x\psi)} &= \sqrt{U_0^2 + \left(\frac{r_0}{2}\right)^2}\, l_\psi\,,\\ \frac{\partial L}{\partial(\partial_x\phi_1)} &= \sqrt{U_0^2 + \left(\frac{r_0}{2}\right)^2}\, l_{\phi_1}\,,\end{aligned} \tag{3.10}$$

while

$$\left(U_0^2 + \left(\frac{r_0}{2}\right)^2\right) R^2\sqrt{1 - f^2(l_\psi, l_{\phi_1})} \tag{3.11}$$

for the energy associated to x, where we have defined

$$f^2(l_\psi, l_{\phi_1}) = \frac{18}{5}(3l_\psi^2 - 2l_\psi l_{\phi_1} + 2l_{\phi_1}^2)\,. \tag{3.12}$$

With the value of the angular momenta fixed through (3.10) and (3.11), and (3.9) for the energy, and using the fact that the minimum is located at $\sqrt{U_0^2 + (r_0/2)^2}$, the solution of these equations for x, ψ and ϕ_1, in terms of U is

$$\begin{aligned}x &= \frac{R^2\sqrt{1 - f^2(l_\psi, l_{\phi_1})}}{\sqrt{U_0^2 + (r_0/2)^2}} \int_1^{U/\sqrt{U_0^2+(r_0/2)^2}} \frac{dy}{y^2\sqrt{(y^2-1)(y^2+1-f^2(l_\psi, l_{\phi_1}))}}\,,\\ \psi &= -\frac{9(3l_\psi - l_{\phi_1})}{10} \int_1^{U/\sqrt{U_0^2+(r_0/2)^2}} \frac{dy}{y^2\sqrt{(y^2-1)(y^2+1-f^2(l_\psi, l_{\phi_1}))}}\\ \phi_1 &= \frac{9(l_\psi - 2l_{\phi_1})}{5} \int_1^{U/\sqrt{U_0^2+(r_0/2)^2}} \frac{dy}{y^2\sqrt{(y^2-1)(y^2+1-f^2(l_\psi, l_{\phi_1}))}}\,.\end{aligned} \tag{3.13}$$

The value of the parameter U_0 can be determined through the condition

$$\frac{L}{2} = x(U = \infty) = \frac{R^2\sqrt{1 - f^2(l_\psi, l_{\phi_1})}}{\sqrt{U_0^2 + (r_0/2)^2}} I_1(l_\psi, l_{\phi_1})\,, \tag{3.14}$$

where, as in [21], $I_1(l_\psi, l_{\phi_1})$ represents an expression that can be calculated in terms of elliptic integrals,

$$I_1(l_\psi, l_{\phi_1}) = \frac{1}{(1-f^2)\sqrt{2-f^2}}\left[(2-f^2)E\left(\frac{\pi}{2}, \sqrt{\frac{1-f^2}{2-f^2}}\right) - F\left(\frac{\pi}{2}, \sqrt{\frac{1-f^2}{2-f^2}}\right)\right]. \tag{3.15}$$

[3]More geometrically, U_0 will correspond to the symmetric height of the triangle.

Similarly, l_ψ and l_{ϕ_1} can be obtained from the conditions

$$\begin{aligned}\frac{\Delta\psi}{2} &= \psi(U=\infty) = -\frac{9(3l_\psi - l_{\phi_1})}{10} I_2(l_\psi, l_{\phi_1})\,, \\ \frac{\Delta\phi_1}{2} &= \phi_1(U=\infty) = \frac{9(l_\psi - 2l_{\phi_1})}{5} I_2(l_\psi, l_{\phi_1})\,,\end{aligned} \tag{3.16}$$

where $I_2(l_\psi, l_{\phi_1})$ is the integral

$$I_2(l_\psi, l_{\phi_1}) = \frac{1}{\sqrt{2-f^2}} F\left(\frac{\pi}{2}, \sqrt{\frac{1-f^2}{2-f^2}}\right), \tag{3.17}$$

again as in [21], but now with f defined in (3.12).

Equation (3.14) exhibits the dependence of the radial position of the minimum, $U_{\min}$, in the quark distance, but also of this L in the scale r_0 introduced through Higgs mechanism when splitting the set of branes in the bulk. As the distance r_0 to the smooth point decreases, $r_0 \ll U_0$, from (3.14) we recover an inverse law for the relation between L and U_0, as in the $AdS_5 \times S^5$ Wilson loop constructed in [21]. The only difference with that case, as a consequence of the different background, is the f^2 function (3.12). Note also that the large separation limit, $r_0 \gg U_0$, leads to small values of L, as in the Higgs phase studied in [26].

The energy can be calculated if the solutions (3.13) are introduced in the expression (3.8) for the action. The infinite result, arising from the mass of the W boson corresponding to a string stretching to $U = \infty$, can be regularized introducing some cut-off, so that the energy is simply integrated up to some $U_{\max}$ [21]. When the regularized mass $U_{\max}/2\pi$ of the W boson is substracted, the energy

$$E = \frac{\sqrt{U_0^2 + (r_0/2)^2}}{2\pi} \left[\int_1^\infty dy \left(\frac{y^2}{\sqrt{(y^2-1)(y^2+1-f^2)}} - 1\right) - 1\right] \tag{3.18}$$

leads, once (3.14) is taken into account to introduce the distance L separating the two quarks, to the result

$$E = -\frac{1}{\pi} \frac{(4g_{YM}^2 N)^{1/2}}{L} (1-f^2)^{3/2} I_1^2(l_\psi, l_{\phi_1}). \tag{3.19}$$

Thus, we see that for any value of the distance L, as compared to the scale r_0 measuring the W mass, we obtain a coulombic behaviour for the quark potential. The Coulomb dependence is the same as that obtained in [23, 21] for the unbroken Higgs phase of $N = 4$ theories in the large-N limit. This dependence was also obtained for small values of L in [26] for the Higgs phase of $N = 4$ supersymmetric Yang-Mills, where the breaking to $\mathrm{SU}(N/2) \times \mathrm{SU}(N/2)$ was considered. Note that

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the potential (3.19) is coulombic even in the large-L region, where the energy scale is much smaller than the mass of the W particles, which therefore become irrelevant; in this region, we should then still expect a coulombic dependence.[4]

We thus capture the expected Coulomb behaviour of the interaction energy in the limit of very large and very small quark-antiquark distance L. However, deviation from a Coulomb dependence is expected at intermediate distances. The energy scale is then comparable to the W boson mass which becomes relevant and corrects the interaction of the quarks. This behaviour cannot be obtained from geodesics obeying the symmetric constraint because once (3.3) is imposed in the harmonic function the metric becomes exactly AdS. In order to study deviations from the conformally invariant behaviour indirectly imposed by the symmetric choice (3.3), we should consider more general geodesics. The authors of [26], when studying the Higgs phase of a two center configuration in $N = 4$ supersymetric Yang-Mills, exhibited the validity of their symmetric choice of geodesics by studying small perturbations of their solution. Their approach turns however estremely involved in our case since we are trying to capture the features associated with a non spherical manifold, the T^{11}. However, most of their arguments should still be valid in our case.

In order to exhibit deviations from the Coulomb behaviour arising from non conformality at intermediate energies (or distances), we will calculate the expression for $x = x(U)$ in a configuration different from the symmetric one (the explicit form of this expression, as well as those for $\psi(U)$ and $\phi_1(U)$, can be obtained along the same lines as (3.13)). As the calculation is rather involved, and requires a series expansion, we will chose to concentrate, for simplicity, in geodesics satisfying the radial requirement that both $\vec{U}$ and $\vec{r}_0$ are along the same direction, so that the harmonic function can be written as $H(U) = R^4(U^4 + (U - r_0)^4)/U^4(U - r_0)^4$. Solving (3.9) and (3.10) for $\partial_x U$, we find that now the value of $U_{\min}$ should be determined through the series

$$\frac{L}{2} = \frac{R^2}{U_{\min}}\left[F_0(l_\psi, l_{\phi_1}) + \left(\frac{r_0}{U_{\min}}\right) F_1(l_\psi, l_{\phi_1}) + \mathcal{O}\left(\left(\frac{r_0}{U_{\min}}\right)^2\right)\right], \qquad (3.20)$$

where $F_0(l_\psi, l_{\phi_1})$ and $F_1(l_\psi, l_{\phi_1})$ are integrals that can be calculated in terms of elliptic functions. When $r_0 \ll U_{\min}$, we can drop terms of order $r_0/U_{\min}$ to recover a Coulomb dependence for the interaction energy of the quark-antiquark pair.[5] Besides, from an expansion in $U_{\min}/r_0$ it can be shown that the Coulomb law dependence is also obtained in the opposite limit, where $r_0 \gg U_{\min}$, as expected. However, higher

[4]Large values of L modified the coulombic dependence of the energy in [26]; however, the authors argued the existence of other geodesics, beyond their symmetric choice, where the L^{-1} behaviour should be restored.

[5]Note that we simply present as argument for the deviation from conformality the relation between L and $U_{\min}$. The explicit form of $E = E(L)$ should be obtained graphically, but expression (3.20) is enough to apreciate the origin of the non coulombic dependence.

order terms will correct the conformal behaviour, as in [26]. We thus see that the exact conformal behaviour exhibited by the symmetric configuration is corrected at intermediate distances by other geodesics, as this calculation along the radial direction shows.

Finallly, to conclude the analysis of the solution for the Higgs phase of branes at the conifold in the a symmetric configuration, we note that the value of the energy does not depend on the distance r_0 explicitly, so that expression (3.19) also represents the dependence of the energy in the un-higgsed theory, when $r_0 = 0$ and all D3-branes are at the conifold point. This is also the result obtained in the $U \gg r_0$ limit.

3.3 Analysis of the trajectories

The regularized value for the total energy of the configuration, (3.19), is formally the same as that found in [21] for the case of two quarks at a non constant angle. The missing factor of 2 is a consequence of the two center distribution of branes, constrained to the symmetric condition of section 3.2.[6] However, the requirement of more than one angle to parametrize the Wilson loop on T^{11} is responsible that f^2, entering the energy, becomes now a function of the angular momenta, as defined in (3.12). The vanishing energy of a BPS configuration is shown in [21] to correspond to the limit $\Delta\theta \to \pi$, so that the orientation of the two strings is reversed. Now, the $f^2 = 1$ condition on the angular momenta, associated to the zero values of the energy, becomes the curve

$$\frac{18}{5}(3l_\psi^2 - 2l_\psi l_{\phi_1} + 2l_{\phi_1}^2) = 1\,. \tag{3.21}$$

If we imposse $f^2 = 1$ in (3.16), condition (3.21) can be written in terms of the angles ψ and ϕ_1, parametrizing the loop, as

$$\frac{2}{3}\frac{(\Delta\phi_1)^2}{\pi^2} - \frac{8}{9}\frac{\Delta\phi_1\Delta\psi}{\pi^2} + \frac{16}{9}\frac{(\Delta\psi)^2}{\pi^2} = 1\,, \tag{3.22}$$

which is the equation of a rotated ellipse in the (ψ, ϕ_1) plane. This curve generalizes the condition found in [21], where trajectories associated to a variation $\Delta\theta = \pi$ of the angle measuring the relative orientation of the two strings representing the quarks were shown to correspond to a BPS configuration. Now, all trajectories such that the path where the loop is contained corresponds to a set of values of ψ and ϕ_1 in the ellipse (3.22), are trajectories with vanishing energy. Besides, all trajectories whose variation along the ψ and ϕ_1 directions correspond to points inside the ellipse (points such that $f^2 \leq 1$), including the origin $\psi = \phi_1 = 0$, where no relative angle has been introduced, will lead to real values for the energy.

[6] We have kept this factor explicitly in the definition of the energy in (3.9). Note also the factor 4 in the square root in (3.19), instead of the usual 2, as a consequence of the collection of $2N$ D3-branes that we are considering.

We thus see how the topology of the non spherical horizon T^{11} does not require reversed orientation of the strings, corresponding to $\Delta\theta = \pi$ in the $AdS_5 \times S^5$ background [21].

4. Conclusions

Previous work in the Higgs [26] and Coulomb branches [27] of $N = 4$ SU(N) Yang-Mills dual to $AdS_5 \times S^5$ has proven very fruitful. In particular, continuous distributions of branes [27]–[30] provide examples of renormalization group flow in an AdS/CFT setting [31]–[34]. In the present work we began a program to study similar examples of continuous brane distributions for theories with non spherical horizons [35]. As a first step we have calculated the energy for a quark-antiquark pair in an $\mathrm{SU}(N) \times \mathrm{SU}(N) \times \mathrm{SU}_D(N)$ theory with $N = 1$ supersymmetry using the richer topology of T^{11} background. This Higgs branch corresponds to the flow of the original $\mathrm{SU}(2N) \times \mathrm{SU}(2N)$ field theory under symmetry breaking. We found the expected coulombic behaviour for the quark-antiquark potential, because the harmonic function in the symmetric configuration that we are using still has a quartic dependence on the radial coordinate; thus, we see that the general results obtained in [36] also hold for backgrounds with non spherical horizons. As a by product we also obtain the energy of a quark-antiquark pair for the un-higgsed theory. The energy for the configuration with all branes at the smooth point cannot be obtained from our result, and requires a series expansion. Besides, the relative angle in between the two strings allows the possibility to study states with vanishing energy. For both Higgsed and un-higgsed cases we find that unlike the $AdS_5 \times S^5$ background, there exists a two dimensional curve of BPS states, obtained from different orientation of the quarks, whose existence seems to be a characteristic of non spherical horizons theories. However, the interpretation in the field theory side of these states, obtained from strings with relative orientation different from 0 or π, deserves further study.

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